A numerical representation of acyclic preferences when non-comparability and indifference are concepts with different meaning

LEOBARDO PLATA-PÉREZ[1]

■ **Resumen:** El uso de funciones de utilidad implica transitividad y completitud de las preferencias. La evidencia empírica muestra que estas condiciones no se cumplen en varios casos específicos. Aciclicidad de preferencias representa una debilidad interesante de la racionalidad de preferencias porque admite intransitividad de indiferencias y no comparabilidad entre algunas alternativas. Las correspondencias de utilidad fueron introducidas por Subiza (1994) como una representación numérica de preferencias acíclicas. El marco del análisis de Subiza es modificado en este artículo para obtener representaciones numéricas en las cuales podemos interpretar no-comparabilidad e indiferencia como conceptos con significado distinto. Ambos conceptos han sido considerados con el mismo significado en la presentación típica que asume las preferencias como un concepto primitivo.

■ **Abstract:** The use of utility functions implies transitivity and completeness of preferences. Empirical evidence shows that these conditions fail in several individual cases. Acyclicity of preferences represents an interesting weakness of preference rationality, because it admits intransitivity of indifference and non-comparability between some alternatives. Utility correspondences were introduced by Subiza (1994) as a numerical representations of acyclic preferences. Subiza's framework is modified in this paper in order to obtain numerical representations in which we can interpret non-comparability and indifference as concepts with different meaning. Both concepts have been considered with the same meaning in the standard presentations that assume strict preference as a primitive concept.

1. Universidad Autónoma de San Luis Potosí, Facultad de Economía. Correo electrónico: lplata@uaslp.mx

■ *Introduction*

The numerical representation problem of preferences has been studied in economic theory and mathematical psychology. It has been studied too in general measurement theory.

Numerical representations are used in order to look for preference maximal elements in the set of alternatives. Mathematical optimization tools are applied more easily with numerical representations of preferences. It is possible that the more known example is utility concept.

The use of utility functions implies transitivity and completeness of preferences. This fact is due to the completeness and transitivity of the "greater or equal" numerical relation in the ordered structure of real numbers. Transitivity of indifference relation is implied too assuming transitivity of the weak preference relation.

It is now well-known that choice behavior of individuals in concrete situations there arise strong indications that the transitivity of the individual indifference relation is often violated. It is possible that an individual behavior presents: alternative *x* is indifferent to alternative *y* and alternative *y* is indifferent to alternative z, but however he/she says that alternative *x* is strictly preferred to alternative z[2]. For an adequate description of reality, it is desirable to weak the transitivity of the indifference relation. In this connection it is important to mention the seminal papers of Armostrong (1939, 1950), on which ones assume the existence of certain utility function that express indifference through utility differences. Several authors present some weakness of transitivity of the indifference relation. Other authors present some weakness of transitivity and completeness conditions for a preference relation. We have semi-orderings, interval orderings, partial orderings, quasi-transitivity preferences and acyclic preferences. Pairs of utility functions have been used to obtain numerical representations of semi-orderings and interval orderings. See Luce (1956), Scott *y* Suppes (1958), Jaminson and Lau (1977) and Fishburn (1970, 1985) among others. For partial orderings Fishburn (1970) used the concept of weak utility representation. Herrero *y* Subiza (1992) used representations with numerical subsets for quasi-transitivity preferences. A very interesting concept of utility correspondences is introduced by Subiza (1994) in order to obtain numerical representations of acyclic preferences. The use of these preferences in the theory of consumer demand can be consulted

2. This fact can be occur with preferences like xRy if and only if $.x_1\text{-}y_1. \leq 0.5$, x and y in the R^2_{+}.

in Border (1985) and Uribe (1996). Generalized numerical representations in an interesting unified geometric approach appears in Beja and Gilboa (1992). Numerical representation problem essentially consist in look for an embedding from an empirical structure in a numerical structure. An excellent presentation of these topics appear in Bridges and Mehta (1995).

Acyclicity of preferences is an important case because, in finite domains of alternatives, this condition is equivalent to decisiveness in the sense that an acyclic preference always has maximal elements in a finite domain[3]. For other side, an acyclic preference represents the more general weakened case because it admits intransitivity of indifference and noncomparability between some alternatives. Utility correspondences used by Subiza (1994) in representation of acyclic preferences have the characteristic that each alternative is associated with an upper bounded subset of real numbers. An alternative x is strictly preferred to an alternative y if and only if the supreme of the set associated with x is greater than the supreme of the set associated with y and the intersection of both sets is empty.

Subiza's framework, as many other authors in standard literature, begins given a strict preference P as primitive concept. With P in the base, they usually define indifference relation xIy as not xPy and not yPx. If we admit this, we can confuse indifference concept with non comparability concept. In traditional frameworks the authors force the completeness of preferences but this is not always the case in empirical behavior.

Moreover, they usually use to define weak preference xRy as xPy or xIy, some treatments define xRy as not yPx. In all these frameworks the completeness of preference, xRy or yRx is always a logic law.

In this paper we generalize Subiza's utility correspondence concept in order to have a numerical representation of acyclic preferences that admitting differences between indifference and non comparability concepts. We do this taking as a base for our analysis the weak orders R as primitive concept instead of strict preferences.

■ *Framework and definitions*

Let X be to represent a nonempty set of alternatives. A preference relation R over X is a binary relation $R \subseteq X \times X$. We write xRy instead of $<x, y> \in R$ as usual.

3. See Sen (1970, Chapter 1).

We say that $\langle X, R \rangle$ is a preference structure. Taking R as primitive we define,

xPy	if and only if	xRy and not yRx	(strict preference)
xIy	if and only if	xRy and yRx	(indifference)
$x \sim y$	if and only if not	xRy and not yRx	(non comparability)

We say that structure $\langle X, R \rangle$ is acyclic if for each elements $x_1, x_2, \ldots, x_n$ in X, If $x_1 P x_2, x_2 P x_2, \ldots, x_{n-1} P x_n$ then not $x_n P x_1$.

Definition 1 (Subiza, 1994)

$\mu: X \rightarrow> \Re$ is a utility correspondence for the structure of preferences $<X,P>$ if

a) $\forall\ x \in X$: $\mu(x)$ is a set of real numbers which is nonempty and bounded.

b) $x\,P\,y \leftrightarrow \mu(x) \cap \mu(y) = \phi$ and $\sup \mu(x) > \sup \mu(y)$

We put $<X,P>$ instead of $<X,R>$ because Subiza (1994) works with strict preference P as primitive concept. The set $\Re$ represents the real numbers set. The notation $\mu: X \rightarrow> \Re$ means that μ assigns a subset of real numbers with each alternative x in X. Now we introduce the modified version of above definition with the objective to represent the difference between non comparability and indifference.

Definition 2

$\mu: X \rightarrow> \Re$ is a *R-utility* correspondence for the structure of preferences $<X,R>$ if μ is a utility correspondence which satisfies the following conditions too:

a) $xIy <\!\!-\!\!-\!\!-> \mu(x) \cap \mu(y) \neq \phi$ and $\mu(x) \cap \mu(y) \subset \Re_+$

b) $x \sim y <\!\!-\!\!-\!\!-> \mu(x) \cap \mu(y) \neq \phi$ and $\mu(x) \cap \mu(y) \subset \Re_-$

■ *The result*

Theorem

Let $<X,R>$ be a structure of preferences with X finite or numerable set of alternatives.

Then, $<X,R>$ is acyclic if and only if there exists a R-utility correspondence $\mu : X \rightarrow> \Re$ for the structure of preferences $<X,R>$.

Proof

$\Leftarrow)$

Let be μ a R-utility correspondence for the structure $<X,R>$ and suppose that R contains a cycle. Then we have that there are elements $x_1,x_2,\ldots,x_n$ in X such that

$$x_1 Px_2, \ldots, x_{n-1} Px_n, x_n Px_1.$$

By definition 1 of utility correspondence we have that

$$\sup \mu(x_1) \sup \mu(x_2),\ldots,\sup \mu(x_n) \sup \mu(x_1)$$

Transitivity of > implies that sup $\mu(x_1)$ sup $\mu(x_1)$ and this is a contradiction.

$\Rightarrow)$

Let us consider $X = \{x_i\ x_i\ / \in \aleph\ \}$and $<X,R>$ an acyclic structure of preferences. Using a result of Bridges (1983) we have a function u: $X \rightarrow \Re$ that weakly represents the structure $<X,R>$.

For each $i \in \aleph$ we define numbers a_i and sets A_i and B_i as follows:

$$a_i = \sum_{n \leq i} (1/3^n)$$

$$A_i = \{(a_i + a_k) / x_i \neq x_k \text{ y } x_i\ Ix_k\ \}$$
$$B_i = \{-(a_i + a_k) / x_i \neq x_k \text{ y } x_i \underset{\cdot}{\sim} x_k\}$$

We can define now a R-utility correspondence by

$$\mu(x_i) = A_i \cup B_i \cup \{1 + u(x_i)\}$$

We note that

$$A_i \subset (0,1), B_i \subset (-1,0) \text{ and that } \sup \mu(x_i) = 1 + u(x_i).$$

We must verify that μ is R-utility correspondence for $<X,R>$.
For part (a) we note that $\mu(x_i)$ is nonempty and bounded set because $\mu(x_i) \in \mu(x_i)$ and $\mu(x_i) \subset (-1,2)$.

If we suppose that $x_i\ Px_j$ .we have that sup $\mu(x_i)$sup $\mu(x_j)$.

If . $\mu(x_i) \cap \mu(x_j) \neq \phi$ then $A_i \cap A_j \neq \phi$ or $B_i \cap B_j \neq \phi$ So, it must be happed that $x_i I x_j$ or $x_i \sim x_k$. But this contradicts $x_i P x_j$.

In the same manner, when $\mu(x_i) \cap \mu(x_j) \neq \phi$ and $\sup \mu(x_i) > \sup \mu(x_j)$ we have that $A_i \cap A_j \neq \phi$ or $B_i \cap B_j \neq \phi$ and that $u(x_i) > u(x_j)$
Then we have that not $x_i I x_j$, not $x_i \sim x_k$, and not $x_j P x_i$. So, $x_i P x_j$.
This justifies part (b) of definition.

When $x_i I x_j$. we have that $A_i \cap A_j \neq \phi$.This fact means that $\mu(x_i) \cap \mu(x_j) \neq \phi$. Moreover, $\mu(x_i) \cap \mu(x_j) \cap \Re_+ \neq \phi$ because the number $a_i + a_j$ belongs to the first and second set of the intersection and is positive.

This argument justifies part (a) of definition of *R*-utility correspondence. Part (b) is totally analogously.

We finalize given a simple example. Let us suppose the structure of preferences given by

$X = \{ x_1 , x_2 , x_3 \}$ and $R = \{ \langle x_1 , x_2 \rangle , \langle x_1 , x_3 \rangle \}$. Consider now the correspondences

$\mu (x1) = \{ 3 \}$ $\mu' (x1) = \{ 3 \}$
$\mu (x2) = \{ ½ , 2 \}$ $\mu' (x2) = \{ ½ , 1 \}$
$\mu (x3) = \{ ½ , 1 \}$ $\mu' (x3) = \{ ½ , 2 \}$

These are two examples of utility correspondences representing the given structure. The invariance problem is left for a further study.

■ *References*

Armstrong, W. E. (1939) "The determinatenes of the Utility Function" in *Economic Journal* 49, pp. 453-467.

—— (1950) "A Note on the Theory of Consumer´s Behavior"; in *Oxford Economic Papers*, 2: 119 - 122.

Beja, A. and I. Gilboa, (1992) "Numerical Representations of Imperfectly Ordered Preferences (A Unified Geometric Exposition)", in *Journal of Mathematical Psychology*, 36: 426 - 449.

Border, K. C. (1985) *Fixed Point Theorems with Applications to Economics and Game Theory*. Cambridge University Press.

Bridges, D. S. and G.B. Mehta (1995) "Representations of Preference Orderings". in *Lectures Notes in Economic and Mathematical Systems*, Vol. 422. Springer.

Fishburn, P. (1970) *Utility Theory of Decision Making*. Wiley, NY

—— (1985) Interval - Orders and Interval Graphs. Wiley, NY

Herrero, C. *y* B. Subiza (1992) “Numerical Representations of Partial Orderings” A Discusion WP - AD 92 - 03. Instituto Valenciano de Investigaciones Económicas.

Jaminson, D. T. and L. J. Lau (1977) “The Nature of Equilibrium with Semiorder Preferences”, in *Econometrica*, 45: 1595 - 1605.

Luce, R. D. (1956) “Semiorders and a Theory of Utility Discrimination”, in *Econometrica*. 24: 178 - 191.

Scott, D. and Suppes, P (1958) “Foundational aspects of Theories of Measurement”, in *Journal of Symbolic Logic*, 23: 113 - 128.

Sen, A (1970) *Collective Choice and Social Welfare*, San Francisco, Holden-Day.

Subiza, B. (1994) “Numerical Representability of Acyclic Preferences”, in *Journal of Mathematical Psychology*, Vol. 38 Num4.

Uribe, P. (1996) “Consumidores con preferencias no ordenadas”, in *Estudios Económicos* 11, pp.55-76.